Indian Statistical Institute Bangalore

Statistics and Mathematics Unit

Complex Analysis

M. Math. I Year – Semestral Examination

Instructor: Shubham Jain

Date: April 22, 2026 **Duration: 3 Hours** **Total Marks: 50**

Name: _____

Note: You may use theorems covered in the course, provided you clearly state them before applying them. .

Q. 1. Determine whether the given statements are **True** or **False**. Give a brief justification for your answer. **[15 Marks]**

(1) Consider the function

$$f(z) = \begin{cases} z, & \text{if } |z| \leq 1, \\ z^2, & \text{if } |z| > 1. \end{cases}$$

Then f is not differentiable on $\mathbb{C}$.

(2) Let f be a function defined on a domain Ω such that f^3 is analytic and the real part of f^3 is constant on Ω. Then f is a constant function.

(3) Let Ω be a domain and $a \in \Omega$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions on Ω such that each f_n has a zero of order n at $z = a$. If f_n converges pointwise to a function f, then $\{f_n\}$ cannot converge normally to f.

(4) Let Ω be a domain and $a \in \Omega$. If for every closed curve γ in $\Omega \setminus \{a\}$,

$$\int_\gamma f(z)\, dz = 0,$$

then f has a removable singularity at $z = a$.

(5) Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function such that $f(0) = 0$ and $|f'(0)| \leq 1$. Then $f(\mathbb{D}) \subseteq \mathbb{D}$.

Q. 2. How many roots of the polynomial

$$P(z) = z^5 + 6z^3 + 2z + 10$$

lie in the annulus $1 < |z| < 3$.

OR

Does there exist an entire function f such that

$$f\left(\frac{1}{n}\right) = \frac{n}{2n-1}, \quad \text{for all } n \in \mathbb{N}.$$

[5 Marks]

Q. 3. Let f be an entire functions such that $f(\mathbb{R}) \subseteq \mathbb{R}$ and $f(i\mathbb{R}) \subseteq i\mathbb{R}$. Then f is odd.
{**Hint:** Consider $g(z) = f(z) + f(-z)$.}

OR

Let f be an entire function satisfying

$$f(z+1) = f(z) \quad \text{and} \quad f(z+i) = f(z) \quad \text{for all } z \in \mathbb{C}.$$

Show that f is bounded. **[5 Marks]**

Q. 4. Let P be a polynomial of degree 2 such that

$$\int_{|z|=2} \frac{zP'(z)}{P(z)} dz = 0 \quad \text{and} \quad \int_{|z|=2} \frac{z^2 P'(z)}{P(z)} dz = -2.$$

Then show that both the roots of P are inside $|z| < 2$. If $P(0) = 2026$ then find $P(z)$.
[7 Marks]

Q. 5. Let $f : \mathbb{D} \to \{z \in \mathbb{C} : e^{-\pi/2} < |z| < e^{\pi/2}\}$ be an analytic map such that $f(0) = 1$. Then
(1) Prove that there exists g such that $e^g = f$ and $g(0) = 0$.
(2) Show that $g(\mathbb{D}) \subseteq \{z \in \mathbb{C} : -\pi/2 < \mathrm{Re}(z) < \pi/2\}$.
(3) Show that the map $h(z) = e^{ig(z)}$ maps $\mathbb{D}$ to right half plane.
(4) Deduce that $|f'(0)| < 2$.

OR

(1) Let $\Omega = \mathbb{C} \setminus \{-1, 0, 1\}$. Find an automorphism of Ω (other than identity) fixing ∞, and another not fixing ∞.
(2) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$ using the residue theorem.

[8 Marks]

Q. 6. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with a zero of order 2 at $z = \frac{1}{4}$. Is it possible that $f(0) = -\frac{1}{16}$? If yes, then find all such functions. **[5 Marks]**
Hint: Apply Schwarz Lemma

Q. 7. Consider

$$\mathcal{F} = \{f : \mathbb{D} \to \mathbb{C} \text{ is analytic} : f(0) = 1 \text{ and } \mathrm{Re}(f) > 0\}.$$

Prove that $\mathcal{F}$ is locally uniformly bounded. **[5 Marks]**